J Math Chem (2014) 52:2532–2545
DOI 10.1007/s10910-014-0398-x

ORIGINAL PAPER

Dynamics of the stochastic low concentration trimolecular chemical reaction model

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Received: 8 April 2014 / Accepted: 25 July 2014 / Published online: 13 August 2014

Abstract In this paper, we will introduce a low concentration trimolecular stochastic chemical reaction model. We will prove that the solution of the system is positive and global. And then we draw a conclusion that there is a stationary distribution for the stochastic system and it has ergodicity under appropriate conditions. Finally, we test our theory conclusion by simulations. It is interesting that no matter what states the unique equilibrium of ordinary differential equation model appears, and regardless of whether the limit cycle exists, our stochastic model is always ergodic.

Keywords Trimolecular chemical reaction · Lyapunov function · Stationary distribution · Ergodicity

1 Introduction

In history, the earliest report of periodic chemical reactions in the homogeneous solution is the decomposition of iodate iodine catalyzed the oxidative coupling reaction of

The work was supported by NSFC of China (No. 11371085), and the Ph.D. Programs Foundation of Ministry of China (No. 200918).

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hydrogen peroxide. The earliest use of mathematical models to predict sustained oscillation is Lotka model. But when the chemical oscillation is regarded as rare examples, even as it is playing skills, no attention. Since the early nineteen sixties, due to the discovery of sustained oscillations in the biochemistry reaction, and they have become an important part of experimental research. At the same time, people pay more and more attention in the field of mathematical model in order to explain the periodic oscillation phenomena such as ecology, chemical reactor function and biological variety. For example, Lotka model, Brussel oscillator, Belousov–Zhabotinskii etc.

In this paper, we will introduce a low concentration trimolecular reaction model which is little different from the famous Brusselator model, its reaction mechanism is:

$$\begin{aligned} &A \rightarrow X \\ &B + X \rightarrow Y \\ &X + 2Y \rightarrow 3Y \\ &Y \rightarrow 0. \end{aligned}$$

Here we assume that all the reaction rate constants are equivalent to 1. In the reaction process A, B are input chemicals held at constant concentrations far from equilibrium, denoted by A and B respectively. X, Y are intermediates whose concentration varies with the reaction time. We record x and y as the concentration of the intermediate product X and Y, which is all we want to discuss in this paper. In the well-stirred case and if stochastic fluctuations are neglected the evolution of the concentration of the intermediate product X and Y can be described by the following nonlinear response equation:

$$\begin{cases} \frac{dx}{dt} = A - Bx - xy^2 \\ \frac{dy}{dt} = Bx + xy^2 - y \end{cases} \tag{1.1}$$

There is one and only one equilibrium $P(\frac{A}{B+A^2}, A)$ of system (1.1). Tyson and Light [1] show that the system executes relaxation oscillations in the limit $B \rightarrow 0$. Linear stability analysis for this system about the steady state $X = \frac{A}{B+A^2}$, $Y = A$ yields by Kauffman and Wille [2]. When $\frac{2A^2}{B+A^2} < B + A^2 + 1$, P is steady focal point or nodal point. When $\frac{2A^2}{B+A^2} > B + A^2 + 1$, P is unstable focal point or nodal point. When $\frac{2A^2}{B+A^2} = B + A^2 + 1$, P is stable fine focus. In [3], authors conclude that when $\frac{2A^2}{B+A^2} > B + A^2 + 1$, system (1.1) exists at least one stable limit cycle surrounding the steady state and discuss the condition of $A < 1$, $B < 0.13$.

The reaction mechanism of this model is similar to Prigogine model [4], for which there is no restriction on the superior limit of parameters. In Prigogine model, it is only require that $A > 0$, $B > 1$. Qualitatively, Prigogine trimolecular model is suitable for high concentration ranges, yet our model is fit to low concentration ranges.

In fact, chemical reaction models are inevitably affected by environmental white noise which is an important component in realism, because it can provide an additional degree of realism in compared to their deterministic counterparts. Recently, we have study a stochastic multi-molecule biochemical reaction model in [5]. It is the first time that we consider the chemical reaction model under stochastic perturbation. The paper introduces the dynamics of a stochastic multi-molecule biochemical reaction model by choosing appropriate Lyapunov function. So both from a chemical and a mathematical perspective, there are different possible approaches to include random effects in the model (1.1). In this paper, taking into account the effect of randomly fluctuating environment in system (1.1), we incorporate white noise in order to modelling stochastic chemical reaction model. We introduce stochastic perturbations of the white noises into the ordinary differential equation model directly, then we obtain the corresponding stochastic three molecular reaction model:

$$\begin{cases} \dot{x} = A - Bx - xy^2 + \sigma_1 x \dot{B}_1(t) \\ \dot{y} = Bx + xy^2 - y + \sigma_2 y \dot{B}_2(t) \end{cases} \tag{1.2}$$

where $B_1(t)$, $B_2(t)$ are independent Brownian motions, and $\sigma_1^2 > 0$, $\sigma_2^2 > 0$ represent the intensities of white noise.

In Sect. 2, we will prove that the solution of system (1.2) is positive and global. In Sect. 3, we have a conclusion that for any initial value $(x(0), y(0)) \in R_+^2$, there is a stationary distribution for system (1.2) and it has ergodic property under appropriate conditions. Finally, we test our theory conclusion by simulations in Sect. 4. In this paper, no matter what state the unique equilibrium of ordinary differential equation model appears, such as stable focal point or nodal point, unstable focal point or nodal point, or stable fine focus, and no matter whether the limit cycle exists, our stochastic model is always has ergodicity. This fact makes us feel very interesting. It seems like the white noise causes the stochastic system stable.

In this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets), unless otherwise specified. Denote

$$R_+^n = \{x \in R^n : x_i > 0 \;\; for\, all\; 1 \leq i \leq n\},$$
$$\bar{R}_+^n = \{x \in R^n : x_i \geq 0 \;\; for\, all\; 1 \leq i \leq n\}.$$

Generally we consider d-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad for\, t \geq t_0, \tag{1.3}$$

with initial value $x(t_0) = x_0 \in R^n$, $B(t)$ denotes d-dimensional standard Brownian motions defined on the above probability space. Define the differential operator L associated with Eq. (1.3) by

$$L = \frac{\partial}{\partial t} + \sum f_i(x, t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum \left[g^T(x, t)g(x, t)\right]_{ij} \frac{\partial^2}{\partial x_i x_j}.$$

If L acts on a function $V \in C^{2,1}(R^n \times \overline{R}_+; \overline{R}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace\left[g^T(x,t)V_{xx}(x,t)g(x,t)\right],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i x_j})_{d\times d}$. By Itô's formula, if $x(t) \in S_h$, then

$$dV(x(t),t) = LV(x(t),t)dt + V_x(x(t),t)g(x(t),t)dB(t).$$

2 Existence and uniqueness of the positive solution

Firstly, we show that the solution of system (1.2) is positive and global. For any initial value to get a unique global solution, i.e, no explosion in a finite time, the coefficients of the equation are required to satisfy the local Lipschitz condition and the linear growth condition (cf. Mao [6]). However as the item xy^2 is nonlinear, so the coefficients of system (1.2) do not satisfy the linear growth condition obviously. Thus the solution may be explore in finite time. In this section, we use the Lyapunov analysis method to show that the solution of system (1.2) is positive and global as mentioned in Refs. [7–9].

Theorem 2.1 *If* σ_1, σ_2 *satisfy*

$$\sigma_1^2 < 2B, \sigma_2^2 < 1 \tag{2.1}$$

then there is a unique solution $(x(t), y(t))$ *of system* (1.2) *on* $t \geq 0$ *for any initial value* $(x(0), y(0)) \in R^2_+$*, and the solution will remain in* R^2_+ *with probability 1, that is,* $(x(t), y(t)) \in R^2_+$ *for all* $t \geq 0$ *almost surely.*

Proof Since the coefficients of stochastic differential equation (1.2) are locally Lipschitz continuous for any given initial value $(x(0), y(0)) \in R^2_+$, there exists a unique local solution $(x(t), y(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (see Ref. [6]). We need to proof that $\tau_e = \infty \; a.s.$ in order to show this solution is global, Let $m_0 \geq 0$ be sufficiently large such that $x(0)$ and $y(0)$ all lie within the interval $[1/m_0, m_0]$. For each $m \geq m_0$, define the stopping time:

$$\tau_m = inf\left\{t \in [0, \tau_e) : min\{x(t), y(t)\} \leq \frac{1}{m} \; or \max\{x(t), y(t)\} \geq m\right\}.$$

Throughout this paper, we set $inf\,\phi = \infty$, as usual ϕ denotes the empty set. According to the definition, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, where $\tau_\infty \leq \tau_e$ a.s. If we can prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(x(t), y(t)) \in R^2_+$ a.s. for all $t \geq 0$. That is to say, to complete the proof all we need to illustrate is that $\tau_\infty = \infty \; a.s.$ If it doesn't hold up, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon, \quad for\ all\ m \geq m_1. \tag{2.2}$$

Define a C^2-function $V : R_+^2 \to \overline{R}_+$ by

$$V(x, y) = (x - 1 - logx) + (y - 1 - logy) + k_1 \frac{(x+y)^2}{2} + k_2 \frac{x^2}{2} := V_1 + k_2 V_2 + k_3 V_3.$$

This function is non-negative for all $x, y > 0$ because of $u - 1 - logu \geq 0, \forall u > 0$. Let $m \geq m_0$ and $T > 0$ be arbitrary, using Itô's formula, we obtain

$$\begin{aligned} dV(x, y) &= LVdt + \sigma_1(x-1)dB_1(t) + \sigma_2(y-1)dB_2(t) \\ &\quad + k_2(x+y)(\sigma_1 x dB_1(t) + \sigma_2 y dB_2(t)) + k_3\sigma_1 x^2 dB_1(t). \end{aligned} \tag{2.3}$$

L is the generating operator of system (1.2) and then we get:

$$\begin{aligned} LV_1 &= \left(1 - \tfrac{1}{x}\right)(A - Bx - xy^2) + \tfrac{\sigma_1^2}{2} + \left(1 - \tfrac{1}{y}\right)(Bx + xy^2 - y) + \tfrac{\sigma_2^2}{2} \\ &= -\tfrac{A}{x} - B\tfrac{x}{y} - y - xy + y^2 + A + B + 1 + \tfrac{\sigma_1^2}{2} + \tfrac{\sigma_2^2}{2} \\ &= -\tfrac{A}{x} - B\tfrac{x}{y} - y - xy + y^2 + H \\ &\leq -y + y^2 + H, \end{aligned} \tag{2.4}$$

where $H = A + B + 1 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}$. Next

$$\begin{aligned} LV_2 &= (x+y)(A-y) + \tfrac{1}{2}(\sigma_1 x + \sigma_2 y)^2 \\ &\leq Ax + Ay - xy - y^2 + \sigma_1^2 x^2 + \sigma_2^2 y^2 \\ &\leq Ax + Ay - y^2 + \sigma_1^2 x^2 + \sigma_2^2 y^2, \end{aligned} \tag{2.5}$$

and then

$$\begin{aligned} LV_3 &= Ax - Bx^2 - x^2y^2 + \tfrac{1}{2}\sigma_1^2 x^2 \\ &\leq Ax - Bx^2 + \tfrac{1}{2}\sigma_1^2 x^2, \end{aligned} \tag{2.6}$$

From the definition of function V and the above three inequalities (2.4)–(2.6), we can obtain

$$\begin{aligned} LV &\leq A(k_2 + k_3)x - \left[k_3\left(B - \frac{\sigma_1^2}{2}\right) - k_2\sigma_1^2\right]x^2 + (k_2 A - 1)y \\ &\quad - [k_2(1 - \sigma_2^2) - 1]y^2 + H. \end{aligned} \tag{2.7}$$

Due to condition (2.1), we can choose appropriate positive constant k_2, k_3 such that the following two inequality

$$k_3\left(B-\frac{\sigma_1^2}{2}\right)-k_2\sigma_1^2>0, k_2(1-\sigma_2^2)-1>0$$

hold at the same time. Thus according to (2.7) we can conclude that $LV \leq C$, where C is a constant. The remainder proof of this theorem follows that in Ji et al. [10]. □

3 Ergodicity

Before the proof of the ergodicity, we present a result which can be found in [11]. The reader can also refer to [12] for details.

Let $X(t)$ be a homogeneous Markov Process in E_l (E_l denotes Euclidean l-space) described by the stochastic equation

$$dX(t)=b(X)dt+\sum_{r=1}^{k}g_r(X)dB_r(t). \tag{3.1}$$

and its diffusion matrix is

$$\Lambda(x)=(\lambda_{ij}(x)),\ (\lambda_{ij}(x))=\sum_{r=1}^{k}g_r^i(x)g_r^j(x).$$

Define the differential operator $\mathcal{L}$ associated with Eq. (3.1) by

$$L=\sum_{k=1}^{l}b_k(x)\frac{\partial}{\partial x_k}+\frac{1}{2}\sum_{k,j=1}^{l}\lambda_{kj}(x)\frac{\partial^2}{\partial x_k\partial x_j}.$$

Lemma 3.1 (See [11]) *Assume that there exists a bounded domain $U \subset E_l$ with regular boundary Γ, having the following properties:*

(B.1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $\Lambda(x)$ is bounded away from zero.

(B.2) If $x \in E_l \backslash U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$.

Then the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt=\int_{E_l}f(x)\mu(dx)\right\}=1,$$

for all $x \in E_l$.

Remark 3.1 The proof is given in [11]. The existence of a stationary distribution with density is given in Theorem 4.1, p. 119 and Lemma 9.4, p. 138. The weak convergence and the ergodicity are obtained in Theorem 5.1, p. 121 and Theorem 7.1, p. 130. To validate (B.1), it is enough to prove that F is uniformly elliptical in any bounded domain D, where $Fu = b(x)u_x + \frac{1}{2}tr(\Lambda(x)u_{xx})$, that is, there is a positive number M such that $\sum_{i,j=1}^{k} \lambda_{ij}(x)\xi_i\xi_j \geq M|\xi|^2$, $x \in \bar{D}$, $\xi \in \mathbb{R}^k$ (see Ref. [13, Chapter 3, p. 103] and Rayleighs principle in [14, Chapter 6, p. 349]). To verify (B.2), it is sufficient to show that there exists some neighborhood U and a non-negative C2-function such that, for any $E_l \backslash U$, LV is negative (for details refer to [15, p. 1163]).

Lemma 3.2 *Let $X(t)$ be a regular temporally homogeneous Markov process in E_l. If $X(t)$ is recurrent relative to some bounded domain U, then it is recurrent relative to any nonempty domain in E_l.*

Remark 3.2 Since the existence of positive solution of model (1.2) has been obtained by Theorem 2.1, it is enough to take $\mathbb{R}_+^2$ as the whole space.

Theorem 3.1 *Assume* (2.1) *satisfy. Then, for any initial value* $(x(0), y(0)) \in R_+^2$, *there is a stationary distribution* $\mu(\cdot)$ *for system* (1.2) *and it has ergodic property.*

Proof To prove this theorem, it is enough for us to verify that (B.1) and (B.2) hold under condition (2.1). First, system (1.2) can be written as the form of the following system:

$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A - Bx - xy^2 \\ Bx + xy^2 - y \end{pmatrix} dt + \begin{pmatrix} \sigma_1 x \\ 0 \end{pmatrix} dB_1(t) + \begin{pmatrix} 0 \\ \sigma_2 y \end{pmatrix} dB_2(t)$$

Here the diffusion matrix is

$$\Lambda(x, y) = \begin{pmatrix} \sigma_1^2 x^2 & 0 \\ 0 & \sigma_2^2 y^2 \end{pmatrix}.$$

Besides there is $M = min\{\sigma_1^2 x^2, \sigma_2^2 y^2, (x, y) \in \bar{U}\} > 0$ such that

$$\sum_{i,j=1}^{2} \lambda_{ij}(x, y)\xi_i\xi_j = \sigma_1^2 x^2 \xi_1^2 + \sigma_2^2 y^2 \xi_2^2 \geq M|\xi|^2, \ all\ (x, y) \in \bar{U}, \xi \in \mathbb{R}^2,$$

which implies condition (B.1) is satisfied
Now we need to check condition (B.2). Therefor we will construct a nonnegative C^2-function V and a closed set $U \in \sum$ (which lies in $\mathbb{R}_+^2$ entirely) such that

$$\sup_{(x,y)\in\mathbb{R}_+^2 \backslash U} \mathcal{L}V(x.y) < 0,$$

which can assure that (B.2) is satisfied. Consider C^2-function $h(x, y)$:

$$h(x, y) = \frac{1}{2}x^2 + C_1\frac{(x + y)^2}{2} + C_2(2x + y) - C_3(\log x + \log y), (x, y) \in \mathbb{R}_+^2.$$

Take C_1, C_2, C_3 be positive constant such that

$$C_1 = \frac{B}{\sigma_1^2} - \frac{1}{2}, C_3 = \left(\frac{B}{\sigma_1^2} - \frac{1}{2}\right)(1-\sigma_2^2),$$
$$C_2 > \max\left\{A\left(\frac{1}{\sigma_1^2} + \frac{1}{2B}\right), A\left(\frac{B}{\sigma_1^2} - \frac{1}{2}\right)\right\}. \tag{3.2}$$

It is not difficult to check that $h(x, y)$ has a unique minimum point $\left(x_0, \frac{C_3 x_0}{C_3 - C_2 x_0 - x_0^2}\right)$, where x_0 is the root of equation

$$(C_1+1)x + \frac{C_1 C_3 x}{C_3 - C_2 x - x^2} + 2C_2 - \frac{C_3}{x} = 0, \quad x \in \left(0, \frac{\sqrt{C_2^2 + 4C_3} - C_2}{2}\right),$$

and

$$\lim_{k\to\infty} \inf_{(x,y)\in\mathbb{R}_+^2 \setminus D_k} h(x, y) = +\infty,$$

where $D_k = (1/k, k) \times (1/k, k)$. And then, we define a C^2-function, which is non-negative, taking the following form:

$$V(x, y) = h(x, y) - h\left(x_0, \frac{C_3 x_0}{C_3 - C_2 x_0 - x_0^2}\right).$$

By direct calculation, we obtain that

$$\begin{aligned}
LV &= Ax - Bx^2 - x^2y^2 + \tfrac{1}{2}{\sigma_1}^2 x^2 + C_1\left[Ax + Ay - xy - y^2 + \tfrac{1}{2}(\sigma_1 x + \sigma_2 y)^2\right] \\
&\quad + C_2(2A - y - Bx - xy^2) \\
&\quad + C_3\left(-\tfrac{A}{x} - B\tfrac{x}{y} - xy + y^2 + B + 1 + \tfrac{\sigma_1^2}{2} + \tfrac{\sigma_2^2}{2}\right) \\
&\le Ax - Bx^2 + \tfrac{1}{2}{\sigma_1}^2 x^2 + C_1(Ax + Ay - y^2 + \sigma_1^2 x^2 + \sigma_2^2 y^2) \\
&\quad + C_2(2A - y - Bx) \\
&\quad + C_3\left(-\tfrac{A}{x} - B\tfrac{x}{y} + y^2 + B + 1 + \tfrac{\sigma_1^2}{2} + \tfrac{\sigma_2^2}{2}\right) \\
&\le -[BC_2 - A(C_1+1)]x - \left(B - \tfrac{1}{2}{\sigma_1}^2 - C_1\sigma_1^2\right)x^2 \\
&\quad - (C_2 - C_1 A)y - (C_1 - C_1\sigma_2^2 - C_3)y^2 \\
&\quad - C_3 A\tfrac{1}{x} - C_3 B\tfrac{x}{y} + 2AC_2 + C_3\left(B + 1 + \tfrac{\sigma_1^2}{2} + \tfrac{\sigma_2^2}{2}\right).
\end{aligned}$$

Due to condition (3.2), we choose given value of C_1 and C_3 such that

$$B - \frac{1}{2}{\sigma_1}^2 - C_1\sigma_1^2 = 0, \quad C_1 - C_1\sigma_2^2 - C_3 = 0,$$

and the range of constant C_2 can guarantee

$$BC_2 - A(C_1 + 1) > 0, C_2 - C_1 A > 0.$$

Thus, if remark $K = 2AC_2 + C_3(B + 1 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2})$, then in term of (2.1) and (3.2) we can get:

$$LV \leq -[BC_2 - A(C_1 + 1)]x - (C_2 - C_1 A)y - C_3 A\frac{1}{x} - C_3 B\frac{x}{y} + K. \quad (3.3)$$

Define a closed set

$$U_{\varepsilon_1,\varepsilon_2} = \left\{(x, y) \in \mathbb{R}_+^2 : \varepsilon_1 \leq x \leq \frac{1}{\varepsilon_1}, \varepsilon_2 \leq y \leq \frac{1}{\varepsilon_2}\right\},$$

where $\varepsilon_1, \varepsilon_2$ are sufficiently small numbers such that

$$K - C_3 A\frac{1}{\varepsilon_1} < -1, \quad (3.4)$$

$$K - C_3 B\frac{\varepsilon_1}{\varepsilon_2} < -1, \quad (3.5)$$

$$K - [BC_2 - A(C_1 + 1)]\frac{1}{\varepsilon_1} < -1, \quad (3.6)$$

$$K - (C_2 - C_1 A)\frac{1}{\varepsilon_2} < -1. \quad (3.7)$$

Furthermore ε_2 is a higher order infinitesimal of ε_1. That is to say, we should choose them as the following form:

$$\varepsilon_1 = \varepsilon_0, \quad \varepsilon_2 = \varepsilon_0^2,$$

where ε_0 is a positive number which can be sufficiently small.
Denote

$$D_{\varepsilon_1,\varepsilon_2}^1 = \left\{(x, y) \in \mathbb{R}_+^2 : 0 < x < \varepsilon_1\right\},$$
$$D_{\varepsilon_1,\varepsilon_2}^2 = \left\{(x, y) \in \mathbb{R}_+^2 : \varepsilon_1 \leq x \leq \tfrac{1}{\varepsilon_1}, 0 < y < \varepsilon_2\right\},$$
$$D_{\varepsilon_1,\varepsilon_2}^3 = \left\{(x, y) \in \mathbb{R}_+^2 : x > \tfrac{1}{\varepsilon_1}\right\},$$
$$D_{\varepsilon_1,\varepsilon_2}^4 = \left\{(x, y) \in \mathbb{R}_+^2 : y > \tfrac{1}{\varepsilon_2}\right\}.$$

Then $\mathbb{R}_+^2 \backslash U_{\varepsilon_1,\varepsilon_2} = D_{\varepsilon_1,\varepsilon_2}^1 \bigcup D_{\varepsilon_1,\varepsilon_2}^2 \bigcup D_{\varepsilon_1,\varepsilon_2}^3 \bigcup D_{\varepsilon_1,\varepsilon_2}^4$. Consequently, for any $(x, y) \in \mathbb{R}_+^2 \backslash U_{\varepsilon_1,\varepsilon_2}$ we discuss the following four cases:

Case 1. For any $(x, y) \in D^1_{\varepsilon_1,\varepsilon_2}$,

$$LV \le -C_3 A \frac{1}{x} + K < -C_3 A \frac{1}{\varepsilon_1} + K.$$

We can get $LV < -1$ on $D^1_{\varepsilon_1,\varepsilon_2}$, in view of (3.3) and (3.4).

Case 2. On $D^2_{\varepsilon_1,\varepsilon_2}$,

$$LV \le -C_3 B \frac{x}{y} + K < -C_3 B \frac{\varepsilon_1}{\varepsilon_2} + K,$$

we can also get $LV < -1$ because of (3.3) and (3.5).

Case 3. When $(x, y) \in D^3_{\varepsilon_1,\varepsilon_2}$,

$$LV \le -[BC_2 - A(C_1 + 1)]x + K < -[BC_2 - A(C_1 + 1)]\frac{1}{\varepsilon_1} + K.$$

Noticing (3.3) and (3.6), we have $LV < -1$ on $D^3_{\varepsilon_1,\varepsilon_2}$.

Case 4. On $D^4_{\varepsilon_1,\varepsilon_2}$,

$$LV \le -(C_2 - C_1 A)y + K < -(C_2 - C_1 A)\frac{1}{\varepsilon_2} + K,$$

which indicates $LV < -1$ in this domain, in virtue of (3.3) and (3.7) hold.

Based on the discuss of the above four kinds of cases, the condition (B.2) in Lemma 3.1 is also satisfied.

Thus we complete the proof of Theorem 3.1. □

4 Simulation

In this section, we will test our theory conclusion by simulations. In this paper, we shall assume that the unit of time is minute and the concentrations of the reactant are measured in units of mol/L min. Examples 4.1–4.3 are the scatter distribution comparison diagram of the ordinary differential equation model (1.1) and stochastic differential equation model (1.2). Examples 4.4–4.6 are the simulations of the stationary distribution histogram. In the following simulations, we all use discretization method and choose $\triangle t = 0.02$.

Example 4.1 Choosing the parameters in the system (1.2) as follows:

$$A = 0.9, \quad B = 0.1, \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.5. \tag{4.1}$$

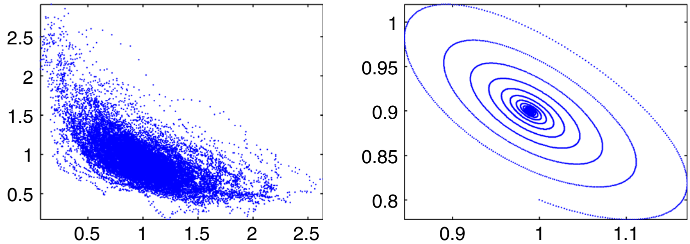

Fig. 1 Computer simulation of the distribution of scatter for stochastic model (1.2) (shown in *left side*) and determined model (1.1) (shown in *right side*) when we choose the value of parameters as (4.1). This moment equilibrium $P(0.989,\ 0.9)$ of system (1.1) is a steady focal point or nodal point

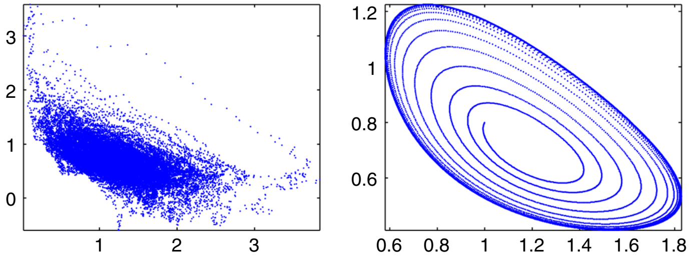

Fig. 2 Computer simulation of the distribution of scatter for stochastic model (1.2) (shown in *left side*) and determined model (1.1) (shown in *right side*) when we choose the value of parameters as (4.2). This moment equilibrium $P(1.18644.\ 0.7)$ of system (1.1) is a unstable focal point or nodal point, and system (1.1) exists unique stable limit cycle

Here we can compute that

$$\frac{2A^2}{B+A^2} \doteq 1.78 < B + A^2 + 1 = 1.91,$$

while the corresponding ordinary differential equation model (1.1) has unique steady focal point or nodal point $P(0.989,\ 0.9)$. Choosing initial value $(x(0), y(0)) = (1, 0.8)$, and then by Matlab, we simulate the scatter distribution of the ordinary differential model (1.1) and the corresponding stochastic model (1.2). The simulation result is shown in Fig. 1.

Example 4.2 Here we set the parameters in the system (1.2) as:

$$A = 0.7, \quad B = 0.1, \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.5. \tag{4.2}$$

We can compute that

$$\frac{2A^2}{B+A^2} \doteq 1.66 > B + A^2 + 1 = 1.59,$$

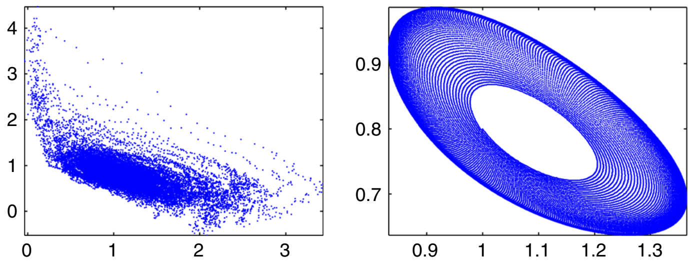

Fig. 3 Computer simulation of the distribution of scatter for stochastic model (1.2) (shown in *left side*) and determined model (1.1) (shown in *right side*) when we choose the value of parameters as (4.3). This moment equilibrium $P(1.09,\ 0.79)$ of system (1.1) is a steady fine focus

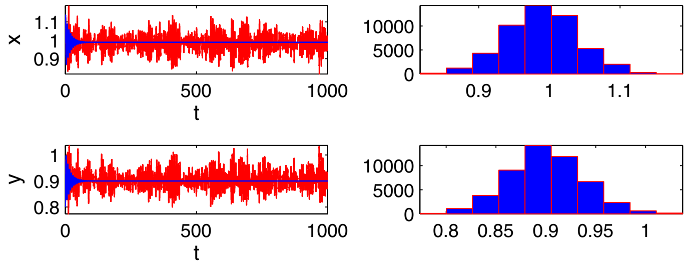

Fig. 4 The solution of the stochastic system and its histogram. The *red lines* represent the solution of system (1.2), and the *blue lines* represent the solution of corresponding undisturbed system (1.1). The pictures on the *right* are the histogram of system (1.2) (Color figure online)

while (1.1) has unique unstable focal point or nodal point $P(1.18644.\ 0.7)$. Keeping initial value same to Example 4.1 and using the same method, we show the simulation result in Fig. 2.

Example 4.3 Now we choose the parameters in the system (1.2) as:

$$A \doteq 0.78968778498212758517, \quad B = 0.1, \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.5, \tag{4.3}$$

such that

$$\frac{2A^2}{B + A^2} = B + A^2 + 1.$$

And then $P(1.09, 0.79)$ is stable fine focus. We show the simulation result in Fig. 3.

Next we will show that system (1.2) has the stationary distribution.

Example 4.4 We choose the parameters in the system (1.2) as:

$$A = 0.9, \quad B = 0.1, \quad \sigma_1 = \sigma_2 = 0.1, \tag{4.4}$$

and so the condition $\sigma_1^2 < 2B, \sigma_2^2 < 1$ is also satisfied. Therefore, as Theorem 3.1 said, there is a stationary distribution (see the histogram on the right in Fig. 4). Furthermore,

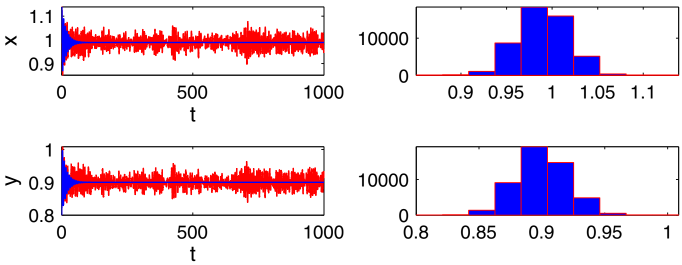

Fig. 5 The solution of the stochastic system and its histogram. The *red lines* represent the solution of system (1.2), and the *blue lines* represent the solution of corresponding undisturbed system (1.1). The pictures on the *right* are the histogram of system (1.2) (Color figure online)

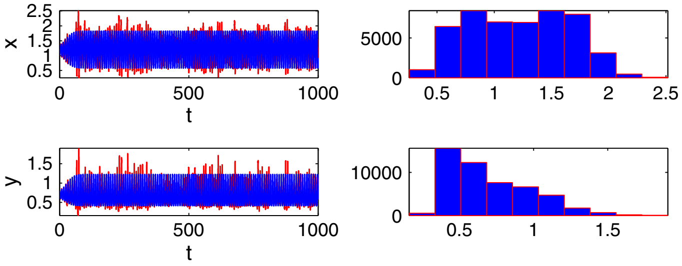

Fig. 6 The solution of the stochastic system and its histogram. The *red lines* represent the solution of system (1.2), and the *blue lines* represent the solution of corresponding undisturbed system (1.1). The pictures on the *right* are the histogram of system (1.2) (Color figure online)

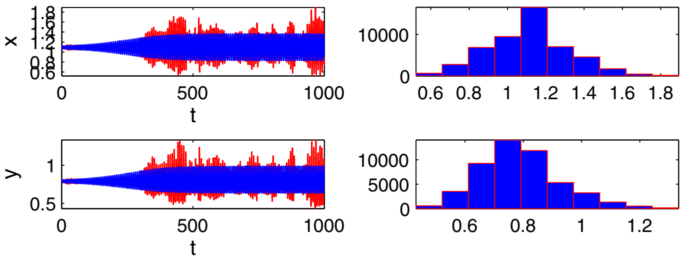

Fig. 7 The solution of the stochastic system and its histogram. The *red lines* represent the solution of system (1.2), and the *blue lines* represent the solution of corresponding undisturbed system (1.1). The pictures on the *right* are the histogram of system (1.2) (Color figure online)

we set

$$A = 0.9, \quad B = 0.1, \quad \sigma_1 = \sigma_2 = 0.05, \tag{4.5}$$

the simulation result of stochastic system is shown in Fig. 5. The left pictures in Figs. 4 and 5 show that the solution of system (1.2) is fluctuating in a small neighborhood, and by comparing the two diagrams we can find that the oscillation amplitude of the stochastic system is reduced with the decrease of white noise.

Example 4.5 Here we set the parameters in the system (1.2) as:

$$A = 0.7, \quad B = 0.1, \quad \sigma_1 = \sigma_2 = 0.1, \tag{4.6}$$

The simulation result of stochastic system is shown in Fig. 6.

Example 4.6 Choosing the parameters in the system (1.2) as

$$A \doteq 0.78968778498212758517, B = 0.1, \sigma_1 = \sigma_2 = 0.1, \tag{4.7}$$

The simulation result of stochastic system is shown in Fig. 7.

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